Appendix　Theoretical Background of Public Finance

Introduction

This chapter provides some key results from consumer demand theory.　It is not meant to be exhaustive but should be viewed as a necessary prelude to studying some aspects of public economics.　Those who want to understand theoretical background fully, are suggested to read basic microeconomic text books such as A Mas-Colell, M.D. Whinston and J.R. Green (1995) *Microeconomic Theory* (Oxford University Press).

Key Definition[1]

1) Weak preference relation or ordering

In the classical theory of consumer demand the basic *weak preference relation* for any consumer i is written as R_i (read as "at least as good as").　If x and y are two consumption baskets available to individual i and if it is xR_iy, then it implies that commodity bundle x is considered by individual i to be at least as good as commodity bundle y.　If at the same time, y is not R_ix then commodity bundle x is *strictly preferred* by individual i to commodity bundle y.　This is written as xP_iy.　If xR_iy and yR_ix simultaneously, then individual i is *indifferent* between commodity bundles x and y.　This is written as xI_iy.

2) Rational weak preference relation

The weak preference relation R_i is *rational* if it possesses the following properties.

(i) Completeness

The ordering R_i must be defined over all pairs of consumption baskets in X_i, that is, for all consumption baskets $a,b \in X_i$, we must have either aR_ib, bR_ia or both (aR_ia is referred to as *reflexivity*).

When comparing two alternatives a and b, it is not necessary to appeal to any third alternative c.　This additional requirement is called as *independence of irrelevant alternative* or *binariness*.

(ii) Forms of transitivity and rationality

Full transitivity requires that if aR_ib and bR_ic then aR_ic.　Typically, to consider an

[1] This part is drawn hearvily from Jha (1998, PartI).

ordering to be rational transitivity usually holds. A weaker form of transitivity is called *quasi-transitivity*. This requires that the strict preference relation is transitive (i.e. gP_ih and hP_ij, then gP_ij).

A still weaker requirement is that of *acyclicity* P_i is acyclic if there does not exist a sequence of consumption baskets $a, b, \cdots\ l$ such that aP_ib and bP_ic and $\cdots\ kP_il$ and lP_ia.

Another requirement is called *non-satiation*, i.e. more is always at least as good as less. A stronger requirement is that of *local non-satiation*. If $a \in X_i$ then these exists another consumption basket u, with $|u - a| < \varepsilon$ with ε arbitrarily small such that uP_ia.

(iii) Convexity

The ordering R_i on X_i is convex if for every $a \in X_i$ the upper contour set is *convex*. In other words, for a, b, c ε X_i, if aR_ib and bR_ic then $[\alpha a + (1-\alpha)b]R_ic$ for $0 \le \alpha \le 1$. *Strong convexity* is defined as for a, b, c $\in X_i$, if aR_ib and bR_ic then $[\alpha a + (1-\alpha)b]P_ic$ for $0 \le \alpha \le 1$. A standard indifference curves satisfy strong convexity.

3) Continuity

The ordering R_i on X_i is continuous if it is presented under limits, i.e. for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^m R_i y^m$ for all n, $x = \lim_{n\to\infty} X^n$ and $y = \lim_{n\to\infty} Y^n$, we have xR_iy.

When an ordering is continuous it can be represented by a real valued utility function such that when xR_iy then $U(x) \ge U(y)$ where $U(1)$ is the real valued utility function[2].

Utility functions[3]

For analytical puposes, utility function is usually assumed to be *twice continuously differentiable*. From the property of convexity of weak preferences we can deduce that $U(\cdot)$ is *quasiconcave*. The utility function $U(\cdot)$ is quasiconcave if the set is convex for all x or, equivalently, if $U(\alpha x + (1-\alpha)y) \ge \min\{U(x), U(y)\}$ for any x, y and all $\alpha \in [0.1]$.

A continuous R_i on $X = R_+^N$ is homothetic if it admit is a utility function $U(x)$ that is *homogeneous of degree one*, i.e. $U(\alpha x) = \alpha U(x)$ for all $\alpha \in [0.1]$.

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2 There is a real valued function $U(\cdot)$ on X_i such that for all X, $x' \in X_i$, xR_ix' implies $U(x) \ge U(x')$ if and only if R is an ordering on X_i and there exists a countable subset of X_i that is P order dense in X_i (Cantor's theorem).

3 This part is drawn hearvily from Jha (1998, PartI).

4) The consumer's utility maximization problem

In the consumer's choice problem, the consumer is assumed to have a rational, continuous and locally non-satiated weak preference. Its utility function is a twice continuously differentiable, quasi-concave function.

The usual set-up of utility maximization problem is defined such that,

$$\max_{x\geq 0} U(x) \quad \text{subject to} \quad p.x \leq y \qquad (1)$$

where y = income, p = price vector.

Marshallian Demand[4]

The Marshallian demand function can be derived from (1.1) such that $x = g(y,p)$. This is the *uncompensated or ordinary demand function*. This demand function possesses the following properties;

(i) *Homogeneity of degree zero* in (y,p), i.e. $X(\alpha y, \alpha p) = X(y, p)$ for any y,p and any scalar $\alpha > 0$.

(ii) *Walras' Law*: $p.x = y$ for all $x \in x(y, p)$

(iii) *Convexity*: If R_i is convex, so that $U(\cdot)$ is quasi-concave, then $x(y,p)$ is a convex set. If R_i is strictly convex, so that $U(\cdot)$ is strictly quasi-concave, then $x(y,p)$ consists of a single element.

If $U(\cdot)$ is continuously differentiable, an optimal consumption bundle $x^* \in X(y, p)$ can be characterized by means of first-order conditions.

$$\partial U(x)/\partial x_i = \lambda p_i \qquad (2)$$

If we have an interior solution then it must be the case that for any two goods r and t:

$$\frac{\partial U(x)/\partial x_r}{\partial U(x)/\partial x_t} = \frac{p_r}{p_t} \qquad (3)$$

The expression on the lefthand side is the marginal rate of substitution of good r for good t. The Lagrange multiplier λ in the first-order condition (1.2) gives the marginal or shadow value of relaxing the constraint in the utility maximization. It equals the consumer's marginal

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4 This part is drawn hearvily from Jha (1998, PartI).

utility at the optimum.

For each $(y,p)>0$, the utility value of the utility maximization problem is denoted as $V(y,p)$. It is equal to $U(x^*)$ for any $x^* \in x(y,p)$. The function $V(y,p)$ is called the *indirect utility function*.

Suppose that $U(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation R_i defined on the consumption set x. The indirect utility function $V(y,p)$ is

(i) homogeneous of degree zero;
(ii) strictly increasing in y and non-increasing in p_k for any commodity k;
(iii) quasiconvex, i.e. the set $\{(y,p):V(y,p)\leq \overline{V}\}$ is convex for any $\overline{V}$;
(iv) continuous in p and y.

The expenditure minimization problem

For $p>0$ and $\overline{U}>0$, minimize p,x subject to $U(\bullet)\geq \overline{U}$.

This problem is to calculate the minimum level of income required to reach the level of utility $\overline{U}$. The expenditure minimization problem is the *dual* to the utility maximization problem. This duality can be expressed in a formal way.

(i) If x^* is optimal in the utility maximization problem when income is $y>0$, then x^* is optimal in the expenditure minimization problem when the required utility level is $U(x^*)$. Moreover, the minimized expenditure level in this expenditure minimization problem is exactly y.
(ii) If x^* is optimal in the expenditure minimization problem. When the required utility level in $\overline{U}>U(0)$, then x^* is optimal in the utility maximization problem when income is p,x^*. Moreover, the maximized utility level in this utility maximization problem is exactly $\overline{U}$.

The expenditure function

Given prices $p>0$ and required utility level $\overline{U}>U(0)$, the value of the expenditure minimization problem is denoted $e(\overline{u},p)$, the expenditure function. It is

(i) homogeneour of degree one in p;
(ii) strictly increasing in U and nondecreasing in p_k for any k;
(iii) concave in p;
(iv) continuous in p and U.

For any $p>0$, $y>0$ and $\overline{U}>U(0)$, it must be the case that;

$e(V(y,p),p)=y$ and that $V(e(p,\overline{U}),p)=\overline{U}$.

This relationship states the equivalence between the conpensating variation and the equivalent variation of a price change.

Hicksian (compensated) Demand[5]

The set of optimal commodity vectors in the expenditure minimization problem is denoted $h(\overline{U},p)$, known as *Hicksian or compensated demand function*. This demand function possesses the following properties;

(i) homogeneity of degree zero in p: i.e. $h(U,\alpha p)=h(U,p)$ for any p, U and $\alpha>0$.

(ii) No excess utility: for any $x\in h(U,p),\ U(x)=U$.

(iii) convexity: if R_i is convex, $h(u,p)$ is a convex set; and if R_i is strictly convex, $U(\bullet)$ is strictly quasi-concave, then there is a unique element in $h(U,p)$.

Hicksian demand and the direction of substitution effects

Along the Hicksian demand function demand changes in the opposite direction of the price change, i.e. for all P_i P_j

$$(P_j-P_i)\left[h(\overline{U},p_j)-h(\overline{U},p_i)\right]\le 0$$

How can we measure the total price change effect when income also changes? The answer can be found in the Slutsky equation

Slutsky equation

$$\begin{aligned}&Hicksian\ demand = Marshallian\ demand\\&h_i(u,p)=x_i(y,p)\end{aligned}\tag{4}$$

Differentiating (1.4) with respect to p_k and evaluating it at $(\overline{p},\overline{u})$, we get

$$\frac{\partial h_i(\overline{p},\overline{u})}{\partial p_k}=\frac{\partial x_i(\overline{p},e(\overline{p},\overline{u}))}{\partial p_k}+\frac{\partial x_i(\overline{p},e(\overline{p},\overline{u}))}{\partial y}\frac{\partial e(\overline{p},\overline{u})}{\partial p_k}$$ as $y=e(\overline{p},\overline{u})$ expenditure function and by definition of duality theorem,

$$\frac{\partial e(\overline{p},\overline{u})}{\partial p_k}=h_k(\overline{p},\overline{u})\tag{5}$$

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5 This part is drawn hearvily from Jha (1998, PartI).

$h_k(\overline{p},\overline{u}) = x_k(\overline{p}, e(\overline{p},\overline{u})) = x_k(\overline{p},\overline{y})$, thus

$$\frac{\partial h_i(\overline{p},\overline{u})}{\partial p_k} = \frac{\partial x_i(\overline{p},\overline{y})}{\partial p_k} + \frac{\partial h_i(\overline{p},\overline{u})}{\partial y} x_k(\overline{p},\overline{y}) \tag{6}$$

The Slutsky equation is, therefore, the partial derivative of Hicksian (compensated) demand for good i with respect to price p_k. In other words, total price effect (along the Marshallian demand function) = substitution effect (along the Hicksian demand function) – income effect.

Shaphard's lemma

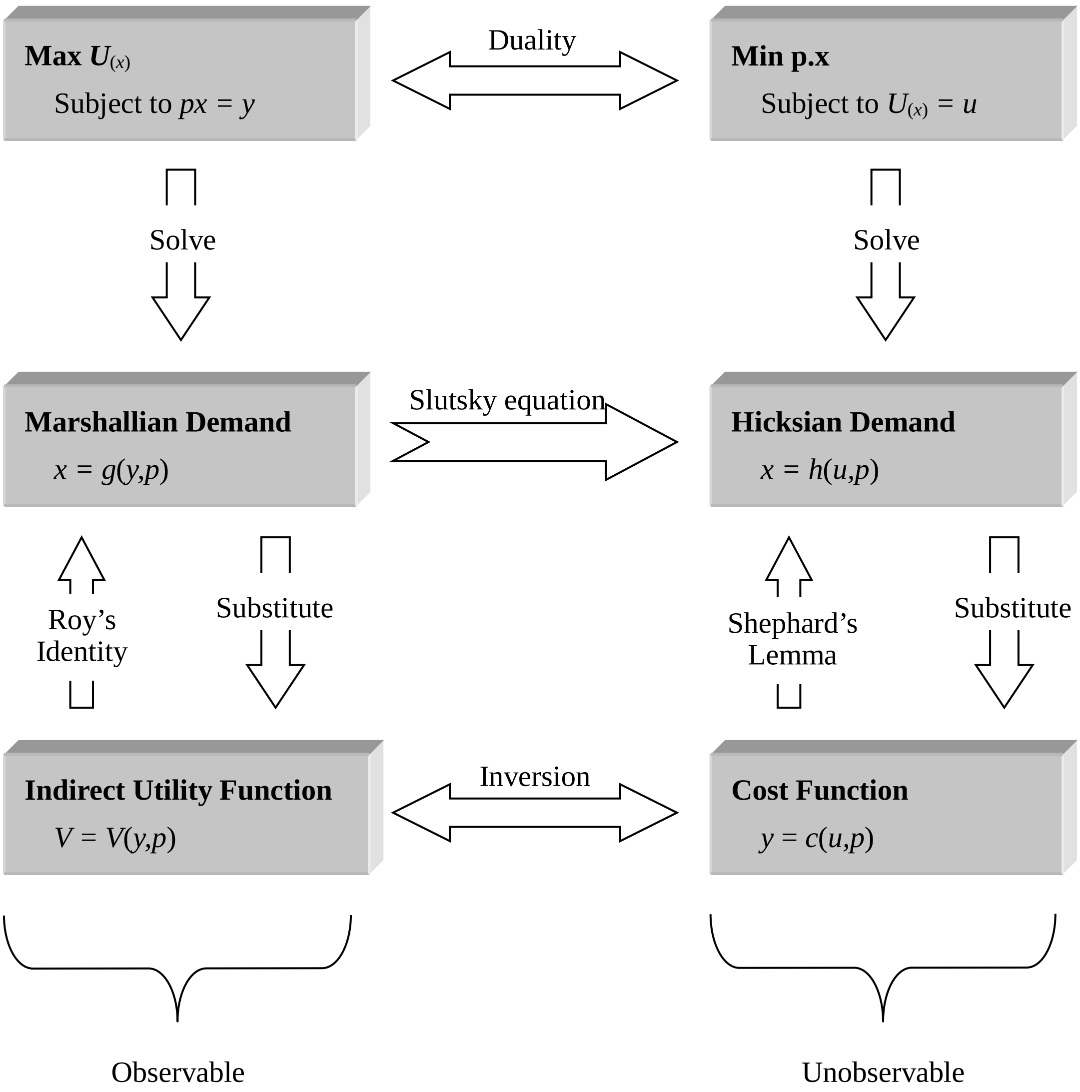
The partial derivative of the cost function with respect to price is the Hicksian (compensated) demand function, that is,

$$\frac{\partial C(u,p)}{\partial p_i} = h_i(p,u) = x_i \tag{7}$$

Roy's Identity

$$x_k = \frac{\partial v / \partial p_k}{\partial v / \partial y} \Rightarrow \frac{\partial v}{\partial p_k} = -\frac{\partial v}{\partial y} x_k = -\Phi x_k \quad \text{where} \quad \Phi = \frac{\partial v}{dy} = \textit{marginal utility of income}$$

Figure 1 Preliminary knowledge of Consumer Demand



Welfare Evaluation of Economic Changes[6]

Suppose that we know the consumer's preferences $\succeq$ and that indirect utility function $v(p,y)$ can be derived from $\succeq$, then it is a simple matter to determine whether the price change makes the consumer better or worse off, depending on the sign of $v(p^1,y)$- $v(p^0,y)$.

In case of welfare change measurement, *money metric* indirect utility functions can be constructed by means of the expenditure function. Starting from any indirect utility function $v(\cdot,\cdot)$, choose an arbitrary price vector $\overline{p} >> 0$, and consider the function $e(\overline{p}, v(p,y))$. This function gives the wealth required to reach the utility level $v(p,y)$ when prices are $\overline{p}$. This expenditure is strictly increasing as a function of the level $v(p,y)$, thus it is an indirect utility function for $\succeq$. $e(\overline{p}, v(p^1,y)) - e(\overline{p}, v(p^0,y))$ provides a measure of the welfare change expressed in money term.

Two natural choices for the price vector $\overline{p}$ are the initial price vector p^0 and the new price vector p^1. These choices lead to two well-known measures of welfare change originating in Hicks (1939), *the equivalent variation* (EV) and *the compensating variation* (CV). Formally, let $u^0 = v(p^0, y)$ and $u^1 = v(p^1, y)$ and note that $e(p^0,u^0) = e(p^1,u^1) = y$, we define

$$EV(p^0, p^1, y) = e(p^0,u^1) - e(p^0,u^0) = e(p^0,u^1) - y \tag{8}$$

and

$$CV(p^0, p^1, y) = e(p^1,u^1) - e(p^1,u^0) = y - e(p^1,u^0) \tag{9}$$

The equivalent variation implies that it is the change in her wealth that would be *equivalent* to the price change in terms of its welfare impact (i.e. the amount of money that the consumer is indifferent about accepting in lieu of the price change). Note that $e(p^0,u^1)$ is the wealth level at which the consumer achieves exactly utility level u^1, the level generated by the price change, at price p^0.

The compensating variation, on the other hand, measures the net revenue of a planner who must *compensate* the consumer for the price change after it occurs, bringing her back to her original utility level u^0.

Figure1.2 depicts the equivalent and compensating variation measures of welfare change.

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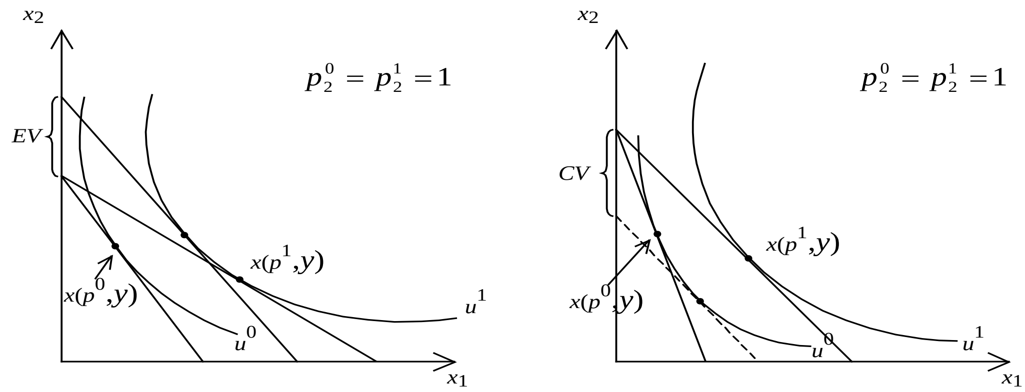
6 This part is drown heavily from Mas-Colell, Whinston and Green (1995), in particular, pages 80-91.

Figure 2　Welfare Evaluation by Utility



(a) Equivalent Variation　　　　(b) Compensating Variation

The equivalent and compensating variations have interesting representations in terms of the Hicksian demand curve.　Suppose, for simplicity, that only the price of good 1 changes, so that $p_1^0 \neq p_1^1$ and $p_l^0 = p_l^1 = \bar{p}_l$ for all $l \neq 1$. Because $w = e(p^0, u^0) = e(p^1, u^1)$ and $h_1(p,u) = \partial e(p,u)/\partial p_1$, we can write

$$\begin{aligned} EV(p^0, p^1, y) &= e(p^0, u^1) - y \\ &= e(p^0, u^1) - e(p^1, u^1) \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1)\,dp_1 \end{aligned} \tag{10}$$

where $\bar{p}_{-1} = (\bar{p}_2, \ldots, \bar{p}_L)$.

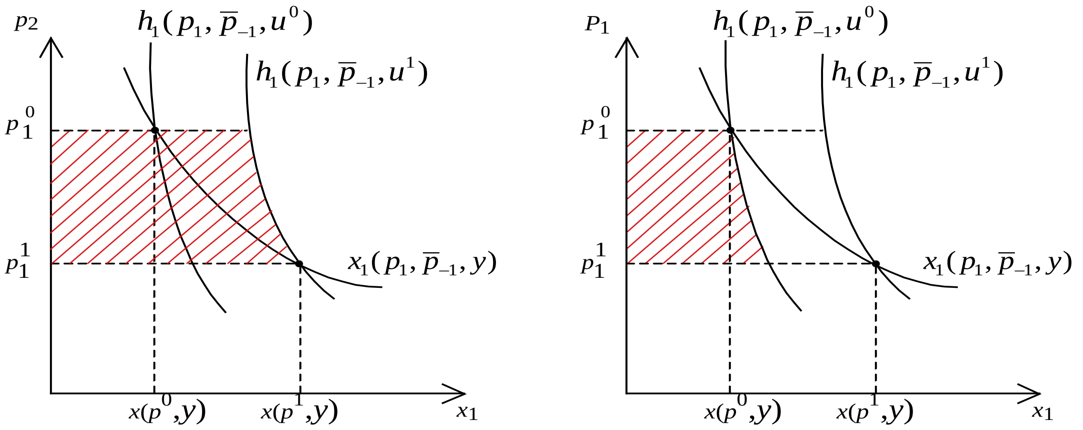
The change in consumer welfare as measured by EV can be represented by the area lying between p_1^0 and p_1^1 and to the left of the Hicksian demand curve for good 1 associated with utility level u^1 (it is equal to this area if $p_1^1 < p_1^0$ and is equal to its negative if $p_1^1 < p_1^0$). The area is depicted as the shaded region in Figure X.2.(a).

Similarly, the compensating variation can be written as

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0)\,dp_1 \tag{11}$$

See Figure 3.(b) for its graphic representation.

Figure 3 Welfare Evaluation by Hicksian Demand



(a) Equivalent Variation (b) Compensating Variation

Figure 3 illustrates a case where good 1 is a normal good. As can be seen in Figure 3, we have $EV(p^0, p^1, y) > CV(p^0, p^1, y)$. This relation between the *EV* and the *CV* reverses when good 1 is inferior. However, if there is no wealth effect for good 1, the *CV* and the *EV* are the same because we have

$$h_1(p, \overline{p}_{-1}, u^0) = x_1(p_1, \overline{p}_{-1}, y) = h_1(p_1, \overline{p}_{-1}, u^1) \tag{12}$$

In absence of wealth effects, the common value of *CV* and *EV* is called as the change in *Marshallian consumer surpurs*.

The Deadweight Loss from Commodity Taxation

Suppose that the government taxes commodity 1, setting a tax on the consumer's purchases of good 1 of *t* per unit. This tax changes the effective price of good 1 to $p_1^1 = p_1^0 + t$ while prices for all other commodities ($l \neq 1$) remain fixed at p_l^1 (so we have $p_l^1 = p_l^0$ for all $l \neq 1$). The total revenue raised by the tax is $T = tx_1(p^1, y)$.

An alternative to this commodity tax that raises the same amount of revenue for the government without changing prices is imposition of a "lump-sum" tax of *T* directly on the consumer's wealth. Is the consumer better or worse off facing this lump-sum wealth tax rather than the commodity tax?

She is worse off under the commodity tax if the equivalent variation of the commodity tax $EV(p^0, p^1, y)$, which is negative, is less than –*T*, the amount of wealth she will lose under the lump-sum tax.

Put in terms of the expenditure function, she is worse off under commodity taxation if

$y - T > e(p^0, u^1)$, so that her wealth after the lump-sum tax is greater than the wealth level that is required at prices p^0 to generate the utility level that she gets under the commodity tax u^1. The difference $(-T) - EV(p^0, p^1, y) = y - T - e(p^0, u^1)$ is known as *the deadweight loss of commodity taxation*. It measures the extra amount by which the consumer is made worse off by commodity taxation above what is necessary to raise the same revenue through a lump-sum tax.

The deadweight loss measure can be represented in terms of the Hicksian demand curve at utility level u^1. Since $T = tx_1(p^1, y) = th_1(p^1, u^1)$, we can write the deadweight loss as follows:

$$
\begin{aligned}
(-T) - EV(p^0, p^1, y) &= e(p^1, u^1) - e(p^0, u^1) - T \\
&= \int_{p_1^0}^{p_1^0 + t} h_1(p_1, \bar{p}_{-1}, u^1)\,dp_1 - th_1(p_1^0 + t, \bar{p}_{-1}, u^1) \qquad (13) \\
&= \int_{p_1^0}^{p_1^0 + t} \left[h_1(p_1, \bar{p}_{-1}, u^1) - h_1(p_1^0 + t, \bar{p}_{-1}, u^1) \right] dp_1
\end{aligned}
$$

Because $h_1(p, u)$ is nonincreasing in p_1, this expression is nonnegative, and it is strictly positive if $h_1(p, u)$ is strictly decreasing in p_1. Figure 4 (a) shows the deadweight loss in the area of the shaded triangular region (*the deadweight loss triangle*).

Figure 4 Deadweight Loss from Commodity Taxation

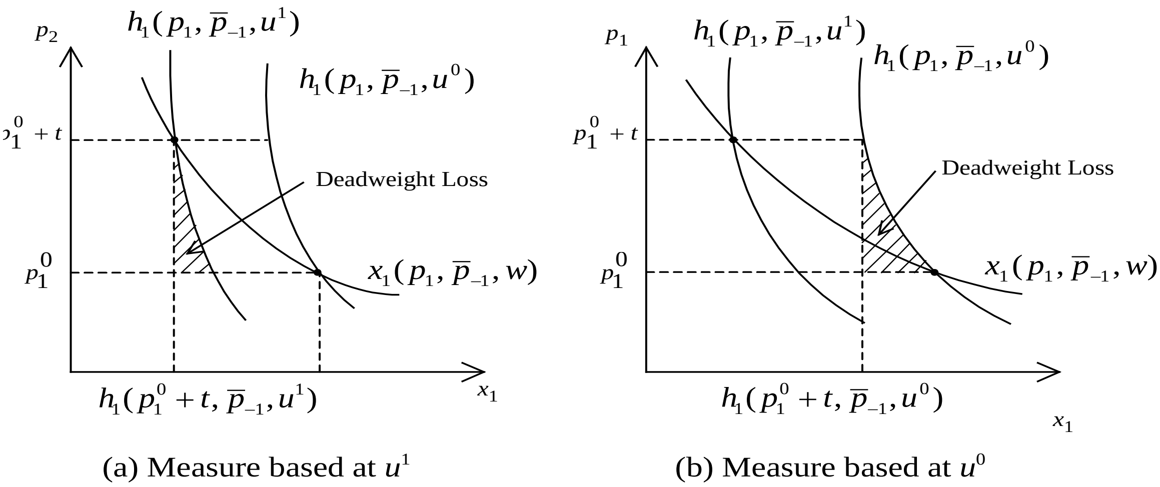
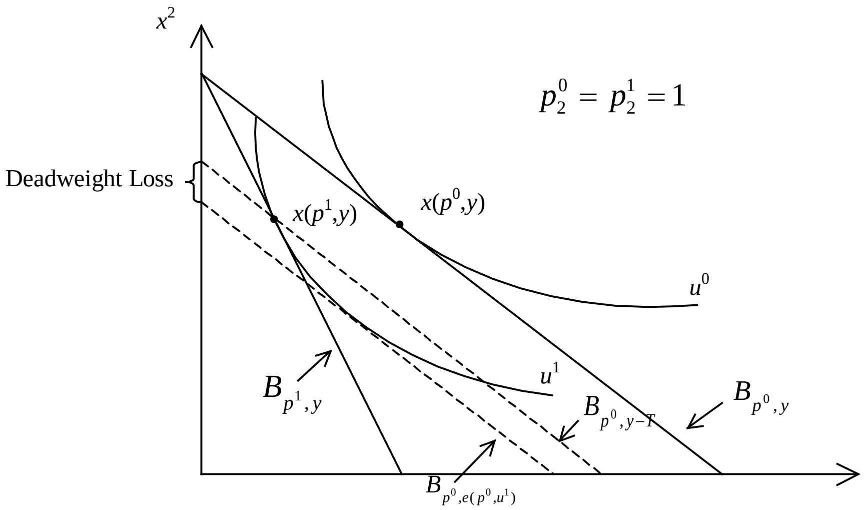


Figure 5　Alternative way to Express Deadweight Loss from Commodity Taxation



Since $(p_1^0 + t)x_1(p^1, y) + p_2^0 x_2(p^1, y) = y$, the bundle $x(p^1, y)$ lies not only on the budget line associated with budget set $B_{p^1,y}$, but also on the budget line associated with budget set $B_{p^0,y-T}$. In contrast, the budget set that generates a utility of u^1 for the consumer at prices p^0 is $B_{p^0,e(p^0,u^1)}$. The deadweight loss is the vertical distance between the budget lines associated with budget sets $B_{p^0,y-T}$ and $B_{p^0,e(p^0,u^1)}$.

A similar deadweight loss triangle can be calculated using the Hicksian demand curve $h_1(p_1,u^0)$. It also measures the loss from commodity taxation, but in a different way.

Suppose that we examine the surplus or deficit that would arise if the government were to compensate the consumer to keep her welfare under the tax equal to her pretax welfare u^0. The government would run a deficit if the tax collected $th_1(p^1,u^0)$ is less than $-CV(p^0, p^1, y)$ or, equivalently, if $th_1(p^1,u^0) < e(p^1,u^1) - y$. Thus, the deficit can be written as

$$\begin{aligned} -CV(p^0, p^1, y) - th_1(p^1,u) &= e(p^1,u^0) - e(p^0,u^0) - th_1(p^1,u^0) \\ &= \int_{p_1^0}^{p_1^0+t} h_1(p^1, \overline{p}_{-1}, u^0)dp_1 - th_1(p_1^0 + t, \overline{p}_{-1}, u^0) \qquad (14) \\ &= \int_{p_0^1}^{p_1^0+t} \left[h_1(p^1, \overline{p}_{-1}, u^0)dp_1 - h_1(p_1^0 + t, \overline{p}_{-1}, u^0)\right]dp_1 \end{aligned}$$

This is strictly positive as long as $h_1(p_1,u)$ is strictly decreasing in p_1. This deadweight loss measure is shown in the shaded area in Figure 4 (b).

Using the Walrasian Demand Curve as An Approximate Welfare Measure

As we have seen above, the welfare change induced by a change in the price of good 1 can be exactly computed by using the area to the left of an appropriate Hicksian demand curve. However the Hicksian demand curve is not directly observable. A simple procedure is to use the Walrasian demand curve instead. We call this estimate of welfare change the *area variation measure* (*AV*):

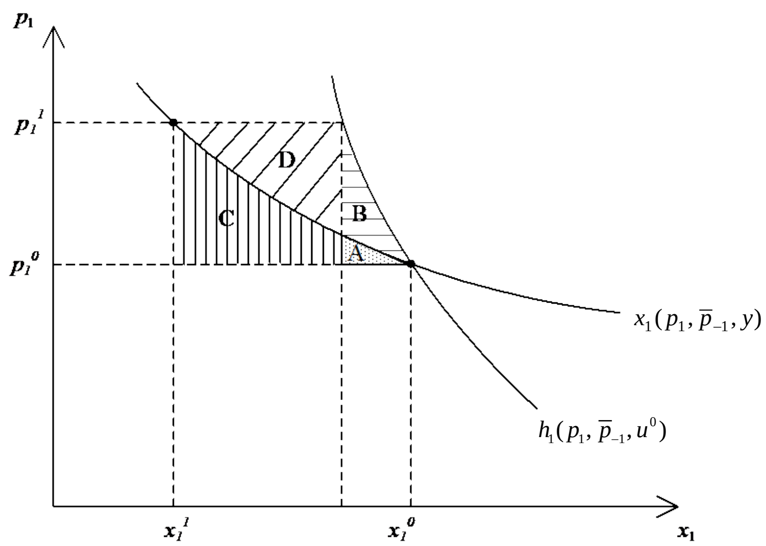
$$AV(p^0, p^1, y) = \int_{p_1^1}^{p_1^0} x_1(p_1, \overline{p}_{-1}, y)dp_1 \tag{15}$$

As Figure 7 (a) and (b) show, when good 1 is normal good, the area variation measure overstates the compensating variation and understates the equivalent variation. When good 1 is inferior, the reverse relations hold. Thus when evaluating the welfare change from a change in prices of several goods, or when comparing two different possible price changes, the area variation measure need not give a correct evaluation of welfare change.

If the wealth effects for the goods under consideration are small, the approximation errors are also small and the area variation measure is almost correct.

If $(p_1^1 - p_1^0)$ is small, then the error involved using the area variation measure becomes small as a fraction of the true welfare change.

Figure 6　Area Variation Measure of Welfare Change



In Figure 7, the area B+D, which measures the difference between the area variation and the true compensating variation becomes small as a fraction of the true compensating variation when $(p_1^1 - p_1^0)$ is small. The area variation measure is a good approximation of the compensating variation measure for small price changes.

However, the approximation error may be quite large as *a fraction of the deadweight loss*. In Figure X.5, the deadweight loss calculated using the Warlasian demand curve is the area A+C, where as the real one is the area A+B. The percentage difference between these two areas need not grow small as the price change grows small.

When $(p_1^1 - p_1^0)$ is small, there is a superior approximation procedure available. Suppose we take a first-order Taylor approximation of $h(p, u^0)$ at p^0,

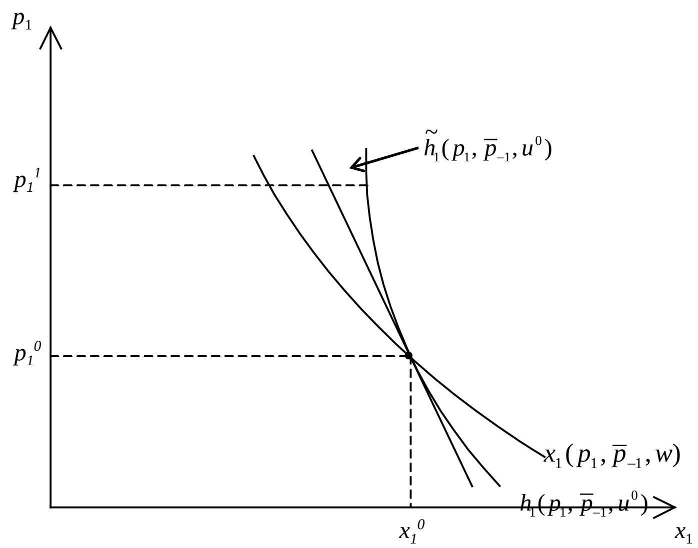
$$\tilde{h}(p, u^0) = h(p^0, u^0) + D_p h(p^0, u^0)(p - p^0) \tag{16}$$

and we calculate

$$\int_{p_1^1}^{p_1^0} \tilde{h}_1(p_1, \bar{p}_{-1}, u^0) dp_1 \tag{17}$$

as an approximation of the welfare change. The function $\tilde{h}_1(p_1, \bar{p}_{-1}, u^0)$ is depicted in Figure 7.

Figure 7 A First-Order Approximation of Demand Function



Because $\tilde{h}_1(p_1, \bar{p}_{-1}, u^0)$ has the same slope as the true Hicksian demand function

$h_1(p,u^0)$ at p^0, for small price changes, this approximation comes closer than expression (2.53) to the true welfare change.

The approximation in (17) is directly computable from knowledge of the observable Walrasian demand function $x^1(p,y)$. To see this, note that because $h(p^0,u^0)=x(g^0,y)$ and $D_p h(p^0,u^0)=s(p^0,w), \tilde{h}(p,u^0)$ can be expressed solely in terms that involve the Walrasian demand function and its derivatives at the point (p^0,y).

$$\tilde{h}(p,u^0)=x(p^0,y)+s(p^0,w)(p-p^0) \tag{18}$$

In particular, since only the price of good 1 is changing, we have

$$\tilde{h}_1(p_1,\bar{p}_{-1},u^0)=x_1(p_1^0,\bar{p}_{-1},y)+s_{11}(p_1^0,\bar{p}_{-1},y)(p_1-p_1^0) \tag{19}$$

where $$s_{11}(p_1^0,\bar{p}_{-1},y)=\frac{\partial x_1(p^0,y)}{\partial p_1}+\frac{\partial x_1(p^0,y)}{\partial w}x_1(p^0,y)$$

When $(p^1- p^0)$ is small, this procedure provides a better approximation to the true compensating variation than does the area variation measure. On the other hand, when $(p^1- p^0)$ is large, it is difficult to judge which is the better approximation.

It is entirely possible for the area variation measure to be superior. After all, its use guarantees some sensitivity of the approximation to demand behavior away from p^0, whereas the use of $\tilde{h}(p,u^0)$ does not.

Fundamental Theorems of Welfare Economics

There are two fundamental theorems showing the equivalence between Pareto optimality and the perfectly competitive market mechanism. Two theorems can be paraphrased as follows:

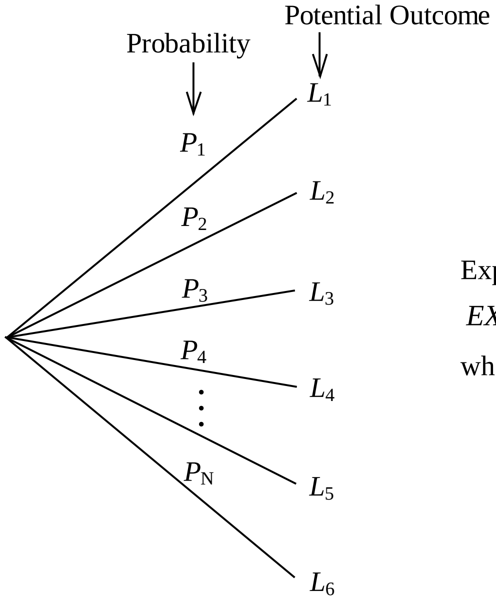
The First Fundamental Welfare Theorem. If every relevant good is traded in a market at publicly known prices, and if households and firms are perfectly competitively, then the market outcome is Pareto optimal. That is, when markets are complete, *any competitive equilibrium is necessarily Pareto optimal* (Formally state that, if the price p^* and allocation $(x_1^*\cdots x_i^*, q_1^*\cdots q_i^*)$ constitute a competitive equilibrium, then this allocation is Pareto optimal). The first welfare theorem provides a set of conditions under which we can be assured that a market economy will achieve Pareto optimal result. To be more precise, by individual maximization behavior, each economic agent responds to prices by equating his marginal rates of substitution for consumers and transformation for firms to these prices. Since all agents face the same prices, all the marginal rates are equated to each other in the equilibrium. Combined with market equilibria, these equalities characterize Pareto optima in a convex environment (i.e. nonincreasing returns for firms and convex preferences for consumers).

The Second Fundamental Welfare Theorem. If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged* (Formally state that, for any Pareto optimal levels of utility ($u_1^* \cdots u_i^*$), there are transfers of the numeraire commodity ($T_1 \cdots T_i$) satisfying $\sum_i T_i = 0$, such that a competitive equilibrium reached from the endowments ($w_{m1} + T_1, \cdots w_{mi} + T_i$) yields precisely a the utility ($u_1^* \cdots u_i^*$). This theorem goes further. It states that under the same set of assumptions as the first welfare theorem plus convexity conditions, all Pareto optimal outcomes can in principle be implemented through the market mechanism. That is, a public authority who wishes to implement a particular Pareto optimal outcome may always do so by appropriately redistributing wealth and then "letting the market work" (Mas-Colell, Whinston, and Green (1995) p.308). In other words, Pareto optimality of the private property competitive equilibrium is satisfactory with respect to the efficiency criterion but it may lead to undesirable income distributions.

Uncertainty

Economists set the choice under uncertainty by considering a situation in which alternatives with uncertain outcomes are describable by means of objectively known probabilities defined on an abstract set of possible outcomes. These representations of risky alternatives are called *lotteries*. Assuming that the decision maker has a rational preference relation over these lotteries, we can construct the expected utility theorem.

A decision maker faces a choice among a number of risky alternatives. Each risky alternative may result in one of a number of possible outcomes, but which outcome will actually occur is uncertain at the time that he must make his choice.



Expected return

$$EX = P_1L_1 + P_2L_2 + \cdots + P_iL_i + \cdots P_NL_N$$

where ΣP=1

An act as a lottery ticket: Consider a ticket that pays \$100 if an odd number is drawn from an urn of 10 equiprobable consequences numbered 1 thorough 10.

How a rational agent evaluates such a lottery? There is a 50% chance of winning \$100. Some might suggest that we use the expected value of the monetary consequences, that is,
0.5×\$100+0.5×\$0=\$50.

Someone would pay a lottery ticket if it is equal to or less than \$50. Consider a repeated coin toss and the lottery that pays \$2 if a head appears for the first time on the n-th toss:

$$U(a) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} 2^n = +\infty$$

According to the expected value criterion, this lottery has infinite value, although no one may be willing to pay \$1000 to play this game. Why?

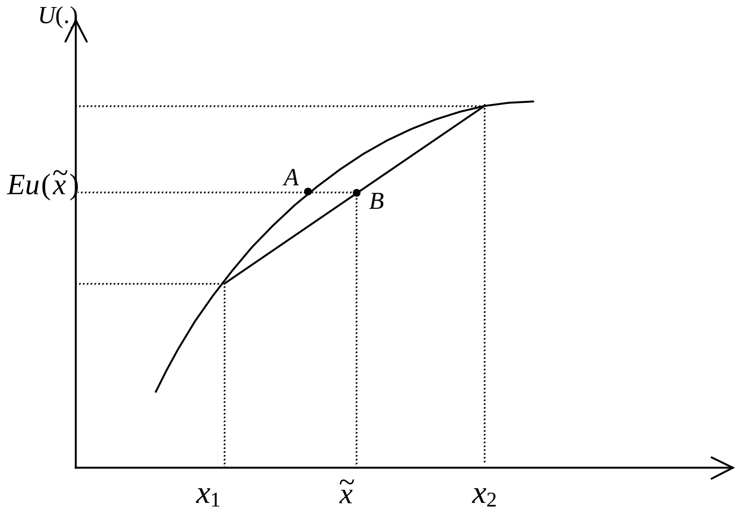
Measures of Risk Aversion

Let $\tilde{x}$ be a lottery defined by $\{x_1 \cdots, x_s; \pi_1, \cdots, \pi_s\}$ so that

$$U(\tilde{x}) = \sum_{S=1}^{S} \pi_s u(x_s) = Eu(x)$$

Concavity of u(.) represents <u>risk aversion</u>.

Figure 8



Concavity implies that

$$\frac{1}{2}u(x_1)+\frac{1}{2}u(x_2)<u\left(\frac{1}{2}x_1+\frac{1}{2}x_2\right)$$

Risk loving is represented by the convexity of u(.). Faced with a lottery ticket that yields x_1 with probability 0.5 and x_2 with probability 0.5, the agent prefers to this lottery a certain return equal to the mean of the returns from the ticket. We define the certainty equivalent of $\tilde{x}$, $EC\tilde{x}$ as the deterministic return that the agent views as equivalent to the stochastic variable $\tilde{x}$, that is,

$$u\big(E(\tilde{x})\big)=Eu(\tilde{x})\,.$$

Then we can define the risk premium associated with $\tilde{x}$, denoted $\rho\tilde{x}$ (i.e. *AB* in Figure 8) by $u\big(E(\tilde{x})\big)-\rho\tilde{x})=Eu(\tilde{x})$.

The Theory of the Second Best

The theory of the second best is concerned with the design of government policy in situations where the economy is characterized by some important distortions that cannot be removed. This is in contrast to "first-best" economies, where all the conditions for Pareto efficiency can be satisfied. Second-best considerations say that it may not be desirable to remove distortions in those sectors where they can be removed. The theory of the second best is often interpreted fallaciously as saying that as long as there are some distortions, economic theory has nothing to say. This is in correct, as we shall shortly show. Economic theory can tell us under what circumstances two small distortions are preferable to one large one; when it is better to have in efficiencies in both consumption and production; and when it is better not to have inefficiencies in production. Second-best theory tells us that we cannot blindly apply the lessons of first-best economics. Finding out what we should do when some distortions exist is often a difficult task, but it is not impossible (Stiglitz, 2000, p.551).

Because of several reasons, e.g. a monopoly in one sector or increasing returns to scale somewhere or something else, Lipsey and Lancaster (1957) proved the following theorem.

The general theorem of second best

(1) If all the conditions for Pareto optimality cannot be met then it is not necessarily second best to satisfy a subset of these conditions;

(2) In general, to attain the second-best optimum it is necessary to violate all the conditions of Pareto optimality.

The actual source of the second-best constraint and why it should be taken seriously are both important issues. In some cases, problems arise because of say a “natural monopoly” or because limp-sum taxer are infeasible or because some distortion has to be maintained for historical reasons. In practice, the most important reason is that decision-aking about public works is often done in isolation of tax policy. No grand coordination of public policy measures is attempted or, it may be argued, is even feasible.

A question that naturally arises is; when is it appropriate to satisfy the Pareto optimum conditions in one sector of the economy irrespective of whether such conditions are satisfied elsewhere. This is the question of when *piecemeal* policy is appropriate. The answer to this question is “quite rarely”.

As a related approach, there is the *second-better* or **n-th** best approach. This approach considers only marginal changes in some distortion and evaluates the welfare consequences of these changes. This method has three practical advantages. First, since only small changes are being evaluated, only local rather than global information is required. Second, there is no need to derive complicated conditions for optimality as in a formal second-best exercise. Third, since only incremental changes are being considered it is possible to derive necessary and sufficient conditions (Jha (1998), pp. 44-46).

Exercises

1. Consider the three good setting in which the consumer has utility function $U(x) = (x_1 - b_1)^{\alpha}(x_2 - b_2)^{\alpha}(x_3 - b_3)$
   (a) Derive the Hicksian demand and expenditure functions.
   (b) Show that the derivatives of the expenditure functions are the Hicksian demand function in (a).
   (c) Verigy that the own-substitution terms are negative and that compensated cross-price effects are symmetric.
2. A consumer in a three-good economy (denoted x_1, x_2 and x_3, prices denoted p_1, p_2 and p_3) with wealth level w>0 has demand functions for commodities 1 and 2 given by

$$x_1 = 100 - 5\frac{p_1}{p_3} + \beta\frac{p_2}{p_3} + \delta\frac{w}{p_3}$$

$$x_2 = \alpha + \beta\frac{p_1}{p_3} + \gamma\frac{p_2}{p_3} + \delta\frac{w}{p_3}$$

   where $\alpha,\ \beta - \gamma\,\delta$ are nonzero constants.
   (a) Indicate how to calculate the demand for good 3 (but do not actually do it).
   (b) Are the demand functions for x, and x_2 appropriately homogeneous?
   (c) Calculate the restrictions on the numerical values of α, β, γ and δ implied by utility maximization.
   (d) Given your results in (c), for a fixed level of x_3 draw the consumer's indifference curve in the (x_1, x_2) plane.
   (e) What does your answer to (d) imply about the form of the consumer's utility function $U(x_1, x_2, x_3,)$?

References

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Laffont, J.J. (1988) *Fundamentals of Public Economics*, translated by Bonin, J.P. and Bonin, H., The MIT Press.

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Appendix 1 Example of actual derivation of Marshallian demand and Hicksian demand

Consider the Cobb-Douglas utility function of the two good economy. The representative households maximizes their utility function , such that

$Max \quad U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ Subject to

$$p_1 x_1 + p_2 x_2 = y \tag{1A.1}$$

By the Lagrangean (after transforming utility function into logarithmic function)

$$L = \alpha \ln x_1 + (1-\alpha)\ln x_2 + \lambda\left[p_1 x_1 + p_2 x_2 - y\right]$$

First order conditions,

$$\frac{\alpha}{x_1} + \lambda p_1 = 0, \quad \frac{1-\alpha}{x_2} + \lambda p_2 = 0 \tag{1A.2}$$

Solve these conditions for x_1, and x_2 , $x_1 = \dfrac{\alpha y}{p_1}$, $x_2 = \dfrac{(1-\alpha)y}{p_2}$. There are, in fact, Marshallian demands, i.e. $x_i = g(y,p_i)$ Substituting the first-order conditions (1A.2) into the constraint $u(h_1(p,u), h_2(p,u))$, we obtain,

$$\begin{aligned} h_1(p,u) &= \left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} u \\ h_2(p,u) &= \left[\frac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} u \end{aligned} \tag{1A.3}$$

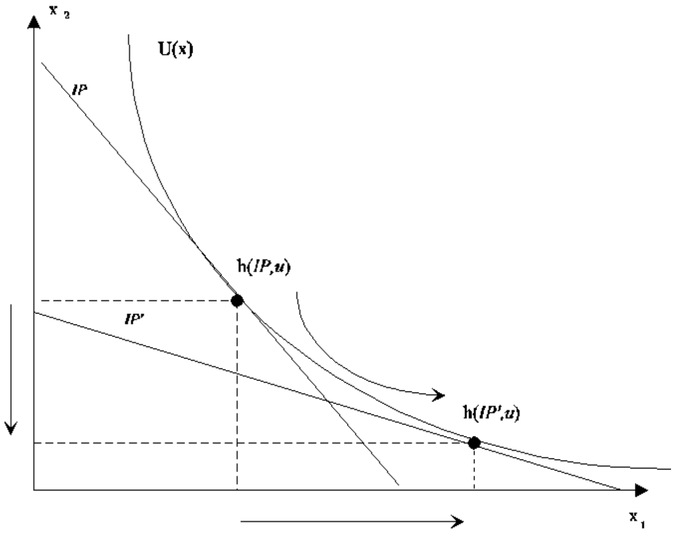
These are, in fact, Hicksian demands, i.e. $h_i = (u,p)$. By definition of expenditure function,

$e(p,u) = \mathbf{p}h(p,u)$ yields

$$e(p,u) = \left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] p_1^{\alpha} p_2^{1-\alpha} u \tag{1A.4}$$

How does the Hicksian (compensated) demand change when the (relative) price vector changes from $\mathbf{p}$ to $\mathbf{p'}$?

Figure 1A.1



<u>Notes:</u>

slope of $IP = \frac{P_1}{P_2}$; slope of $IP^I = \frac{P_1^I}{P_2^I}$

Hicksian (compensated) demand function comes from viewing the demand function as being constructed by varying prices and incomes so as to keep the consumer at a fixed level of utility. Thus, the income changes are arranged to 'compensate' for the price changes. (This is the reason why it is called compensated demand function)

By the Slutsky equation,

$$\frac{\partial h_1}{\partial p_2} = S_{12} = \frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial y}$$

$$\frac{\partial h_2}{\partial p_1} = S_{21} = \frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial y}$$

From (1A.4) $\frac{\partial h_1}{\partial p_2} = \frac{\alpha}{p_1}\left(\frac{1-\alpha}{p_2}\right)y = \frac{\partial h_2}{\partial p_1}$, thus $S_{12} = S_{21} < 0$ (Symmetry).

The properties of demand functions

(1) The substitution term is negative semidefinite

$$\frac{\partial h_j(p,u)}{\partial p_i} = \frac{\partial^2 e(p,u)}{\partial p_i \partial p_j} \leq 0 \qquad \text{\{from (1.5)\}}$$

(2) The substitution term is symmetric

$$\frac{\partial h_j(p,u)}{\partial p_i} = \frac{\partial^2 e(p,u)}{\partial p_i \partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_j \partial p_j} = \frac{\partial h_j(p,u)}{\partial p_j}$$

(3) The compensated own price effect is nonpositive

$$\frac{\partial h_i(p,u)}{\partial p_i} = \frac{\partial^2 e(p,u)}{\partial {p_i}^2} \leq 0$$